Fused Mackey functors

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Abstract: Let G be a finite group. In [5], Hambleton, Taylor and Williams have considered the question of comparing Mackey functors for G and biset functors defined on subgroups of G and bifree bisets as morphisms.

This paper proposes a different approach to this problem, from the point of view of various categories of G-sets. In particular, the category G-$\underline{\mathbf{set}}$ of *fused G-sets* is introduced, as well as the category $\underline{\mathbf{S}}(G)$ of spans in G-$\underline{\mathbf{set}}$. The *fused Mackey functors* for G over a commutative ring R are defined as R-linear functors from $R\,\underline{\mathbf{S}}(G)$ to R-modules. They form an abelian subcategory $\mathsf{Mack}_R^f(G)$ of the category of Mackey functors for G over R. The category $\mathsf{Mack}_{\mathbb{Z}}^f(G)$ is equivalent to the category of conjugation Mackey functors of [5]. The category $\mathsf{Mack}_R^f(G)$ is also equivalent to the category of modules over the *fused Mackey algebra* $\mu_R^f(G)$, which is a quotient of the usual Mackey algebra $\mu_R(G)$ of G over R.

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1. Introduction

This note is devoted to the frequently asked question of comparing Mackey functors for a single finite group G with biset functors defined only on subgroups of G and left-right free bisets as morphisms. The answer to this question has already been given by Hambleton, Taylor and Williams ([5]), but in a rather computational and non canonical way (in particular, in Section 7, the definition of the functor $j_\bullet$ requires the choice of sets of representatives of orbits of any finite G-set).

The present paper makes a systematic use of Dress definition ([3]) and Lindner definition ([6]) of Mackey functors, to avoid these non canonical choices. This leads to the definition of the category of *fused G-sets* (Section 3), and the category of *fused Mackey functors* (Section 4) for a finite group G, which is equivalent to the category of "conjugation invariant Mackey functors" of [5]. This category is also equivalent to the category of modules over the *fused Mackey algebra*, introduced in Section 5.

2. Conjugation bisets revisited

2.1. First a notation : when G is a finite group, and X is a finite G-set, let G-$\mathbf{set}{\downarrow_X}$ denote the category of (finite) G-sets over X: its objects are pairs (Y,b) consisting of a finite G-set Y, and a morphism of G-sets $b:Y\to X$. A

morphism $f : (Y, b) \to (Z, c)$ in G-$\mathbf{set}{\downarrow_X}$ is a morphism of G-sets $f : Y \to Z$ such that $c \circ f = b$.

There is an obvious notion of disjoint union in G-$\mathbf{set}{\downarrow_X}$, and the corresponding Grothendieck group is called the Burnside group over X. It will be denoted by $\mathcal{B}({}_GX)$, or $\mathcal{B}(X)$ when G is clear from the context.

Similarly, when G and H are finite groups, and U is a (G, H)-biset, one can define the category (G, H)-$\mathbf{biset}{\downarrow_U}$ of (G, H)-bisets over U, and the Burnside group $\mathcal{B}({}_GU_H)$ of (G, H)-bisets over U.

2.2. When H is a subgroup of G, and Y is an H-set, induction from H-sets to G-sets is an equivalence of categories from H-$\mathbf{set}{\downarrow_Y}$ to G-$\mathbf{set}{\downarrow_{\mathrm{Ind}_H^G Y}}$. A quasi-inverse equivalence is the functor sending the G-set (X, a) over $\mathrm{Ind}_H^G Y$ to the H-set $a^{-1}(1 \times_H Y)$ (see [2] Lemma 2.4.1). In particular $\mathcal{B}({}_HY) \cong \mathcal{B}({}_G\mathrm{Ind}_H^G Y)$.

2.3. Now an observation: when H and K are subgroups of G, the conjugation (K, H)-bisets defined in Section 6 of [5] are exactly those over the biset ${}_KG_H$ (the set G on which K and H act by multiplication), i.e. the (K, H)-bisets U for which there exists a biset morphism $U \to {}_KG_H$.

Indeed, a conjugation (K, H)-biset U is a bifree (K, H)-biset isomorphic to a disjoint union of bisets of the form $(K \times H)/S$, where S is a subgroup of $K \times H$ of the form

$$S_{g,A} = \{({}^g x, x) \mid x \in A\}$$

where A is a subgroup of H, and g is an element of G such that ${}^gA \le K$. For such a transitive biset $(K \times H)/S$, the map

$$\forall (k, h)S \in (K \times H)/S,\ (k, h)S \mapsto kgh^{-1}$$

is a morphism of (K, H)-bisets.

Conversely, let U be a (K, H)-biset for which there exists a biset morphism $\alpha : U \to {}_KG_H$. Then for any $u \in U$, the stabilizer S_u of u in $K \times H$ is the subgroup

$$S_u = \{(k, h) \in K \times H \mid k \cdot u \cdot h^{-1} = u\}$$

of $K \times H$. Then if $(k, h) \in S_u$,

$$\alpha(k \cdot u) = k\alpha(u) = \alpha(u \cdot h) = \alpha(u)h\ .$$

Let A_u denote the projection of S_u into H, and set $g_u = \alpha(u)$. It follows that $S_u \subseteq S_{g_u, A_u}$.

Conversely, if $(k,h) \in S_{g_u,A_u}$, then $k = {}^{g_u}h$, and there exists some $x \in K$ such that $(x,h) \in S_u$, since $h \in A_u$. Thus $x \cdot u \cdot h^{-1} = u$, from which follows that

$$\alpha(x \cdot u) = xg_u = \alpha(u \cdot h) = g_u h \ \ ,$$

hence $x = {}^{g_u}h = k$, and $S_u = S_{g_u,A_u}$. Observation 2.3 follows.

2.4. In other words, conjugation (K,H)-bisets form a category $\mathbf{Conj}^G_{K,H}$, and there is a forgetful functor $\Phi : (K,H)\text{-}\mathbf{biset}\!\downarrow_{_KG_H} \to \mathbf{Conj}^G_{K,H}$ sending (U,a) to U. This functor is full, preserves disjoint unions, and moreover it induces a surjection on the corresponding sets of isomorphism classes. This means that Φ induces a surjective group homomorphism (still denoted by Φ) from $\mathcal{B}({}_KG_H)$ to the Grothendieck group $\mathcal{B}^G_{K,H}$ of conjugation (K,H)-bisets.

2.5. If H, K and L are subgroups of G, if (U,a) is a (K,H)-biset over ${}_KG_H$ and (V,b) is an (L,K)-biset over ${}_LG_K$, the composition $(V,b)\circ(U,a)$ is the (L,H)-biset over ${}_LG_H$ defined by the following diagram:

$$\begin{array}{ccccc} V & & U & & V\times_K U \\ \big\downarrow{\scriptstyle b} & \circ & \big\downarrow{\scriptstyle a} & = & \big\downarrow{\scriptstyle b\times_K a} \\ {}_LG_K & & {}_KG_H & & G\times_K G \\ & & & & \big\downarrow{\scriptstyle \mu} \\ & & & & {}_LG_H \end{array}$$

where μ is multiplication in G. This composition is associative, and additive with respect to disjoint unions. Hence it induces a composition

$$\widehat{\circ} : \mathcal{B}({}_LG_K)\times\mathcal{B}({}_KG_H) \to \mathcal{B}({}_LG_H) \ \ .$$

Hence, one can define a category $\widehat{\mathbf{B}}(G)$ whose objects are the subgroups of G, and such that $\mathrm{Hom}_{\widehat{\mathbf{B}}(G)}(H,K) = \mathcal{B}({}_KG_H)$, for subgroups H and K of G. Composition is given by $\widehat{\circ}$, and the identity morphism of the subgroup H of G in the category $\widehat{\mathbf{B}}(G)$ is the class of the biset $({}_HH_H, i_H)$, where $i_H : {}_HH_H \to {}_HG_H$ is the inclusion map from H to G.

Since the functor Φ maps the composition $\widehat{\circ}$ to the composition of bisets, and the identity morphism of H in $\widehat{\mathbf{B}}(G)$ to the identity biset ${}_HH_H$, one can extend Φ to a functor $\widehat{\mathbf{B}}(G) \to \mathbf{B}(G)$, which is the identity on objects.

In other words, the category $\mathbf{B}(G)$ introduced in Section 3 of [5] is the quotient of the category $\widehat{\mathbf{B}}(G)$ obtained by identifying morphisms which have the same image by Φ.

2.6. By the above Remark 2.2, when H and K are subgroups of G, there is

a group isomorphism

$$\mathcal{B}({}_KG_H) \cong \mathcal{B}\big(\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H)\big) \enspace ,$$

(with the usual identification of (K,H)-bisets with $(K\times H)$-sets). Now the biset ${}_KG_H$ is actually the restriction to $(K\times H)$ of the (G,G)-biset G. By the Frobenius reciprocity, it follows that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong \mathrm{Ind}_{K\times H}^{G\times G}\mathrm{Res}_{K\times H}^{G\times G}({}_GG_G) \cong \big(\mathrm{Ind}_{K\times H}^{G\times G}\bullet\big) \times {}_GG_G \enspace ,$$

where $\bullet$ is a set of cardinality 1. Since $\mathrm{Ind}_{K\times H}^{G\times G}\bullet \cong (G/K)\times(G/H)$, it follows (after switching G/H and G) that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong (G/K)\times G\times (G/H) \enspace ,$$

where the (G,G)-biset structure of the right hand side is given by

$$\forall (a,b,x,y,g)\in G^5,\;\; a\cdot(xK,g,yH)\cdot b = (axK, agb, b^{-1}yH) \enspace .$$

2.7. It should now be clear that the additive completion $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category whose objects are finite G-sets, where for any two finite G-sets X and Y

$$\mathrm{Hom}_{\mathbf{B}_\bullet(G)}(X,Y) = \mathcal{B}\big({}_G(Y\times G\times X)_G\big) \enspace ,$$

the (G,G)-biset structure on $(Y\times G\times X)$ being given as above by

$$\forall (a,b,g,x,y)\in G^3\times X\times Y,\;\; a\cdot(y,g,x)\cdot b = (ay, agb, b^{-1}x) \enspace .$$

Keeping track of the composition $\widehat{\circ}$ along the above isomorphism shows that the composition in the category $\widehat{\mathbf{B}}_\bullet(G)$ can be defined by linearity from the following: if X, Y, and Z are finite G-sets, if

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[d, "e"] \arrow[dr, "d"] & \\ Z & G & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & U \arrow[dl, "c"'] \arrow[d, "b"] \arrow[dr, "a"] & \\ Y & G & X \end{tikzcd}$$

are (G,G)-bisets over $(Z\times G\times Y)$ and $(Y\times G\times X)$, respectively, their composition is given by the following (G,G)-biset over $(Z\times G\times X)$

$$\begin{tikzcd} & (V\times_{d,c} U)/G \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \arrow[dr, "\alpha"] & \\ Z & G & X \end{tikzcd}$$

where $V \times_{d,c} U$ is the pullback of V and U over Y, i.e. the set of pairs $(v,u) \in V \times U$ with $d(v) = c(u)$, and $(V \times_{d,c} U)/G$ the set of orbits of G on it for the action given by $(v,u) \cdot g = (vg, g^{-1}u)$. This makes sense because $d(v \cdot g) = g^{-1}d(v) = g^{-1}c(u) = c(g^{-1} \cdot u)$ if $d(v) = c(u)$. The map (γ, β, α) is given by

$$(\gamma, \beta, \alpha)\big((v,u)G\big) = \big(f(v), e(v)b(u), a(u)\big) \ .$$

2.8. The functor $\Phi : \widehat{\mathbf{B}}(G) \to \mathbf{B}(G)$ extends uniquely to an additive functor $\Phi_\bullet : \widehat{\mathbf{B}}_\bullet(G) \to \mathbf{B}_\bullet(G)$, and the category $\mathbf{B}_\bullet(G)$ is the quotient of $\widehat{\mathbf{B}}_\bullet(G)$ obtained by identifying morphisms which have the same image by $\Phi_\bullet$. Clearly, two morphisms $f, g \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X,Y)$ are identified if and only if $f - g$ is in the kernel of the group homomorphism

$$\phi : \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \to \mathcal{B}\big({}_G(Y \times X)_G\big)$$

induced by the correspondence

$$\begin{tikzcd} & U \arrow[dl, "c"'] \arrow[d, "b"] \arrow[dr, "a"] & \\ Y & G & X \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & U \arrow[dl, "c"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}$$

on bisets. In other words, a morphism f in $\widehat{\mathbf{B}}_\bullet(G)$ gives the zero morphism in $\mathbf{B}_\bullet(G)$ if and only if it belongs to $\mathrm{Ker}\,\phi$.

2.9. Now the (G,G)-biset ${}_GG_G$ is isomorphic to $\mathrm{Ind}_{\Delta(G)}^{G\times G}\bullet$, where $\Delta(G)$ is the diagonal subgroup of $G \times G$. It follows that there is an isomorphism of (G,G)-bisets

$$Y \times G \times X \cong \mathrm{Ind}_{\Delta(G)}^{G\times G}(Y \times X) \ .$$

Hence, by Remark 2.2 again, since $\Delta(G) \cong G$,

$$\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big) \ ,$$

where ${}_G(Y \times X)$ is the usual cartesian product with diagonal G-action. More precisely, this isomorphism is induced by the correspondence

$$\begin{tikzcd} & U \arrow[dl, "c"'] \arrow[d, "b"] \arrow[dr, "a"] & \\ Y & G & X \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & b^{-1}(1) \arrow[dl, "c"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}$$

It is then easy to check that the composition of

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[d, "e"] \arrow[dr, "d"] & \\ Z & G & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & U \arrow[dl, "c"'] \arrow[d, "b"] \arrow[dr, "a"] & \\ Y & G & X \end{tikzcd}$$

corresponds to the usual pullback diagram

$$\begin{tikzcd} & & e^{-1}(1) \times_{d,c} b^{-1}(1) \arrow[dl] \arrow[dr] & & \\ & e^{-1}(1) \arrow[dl, "f"'] \arrow[dr, "d"] & & b^{-1}(1) \arrow[dl, "c"'] \arrow[dr, "a"] & \\ Z & & Y & & X \end{tikzcd}$$

In other words, the category $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category $\mathbf{S}(G)$ whose objects are the finite G-sets, where

$$\mathrm{Hom}_{\mathbf{S}(G)}(X,Y) = \mathcal{B}\big({}_G(Y \times X)\big) \ ,$$

and composition is induced by pullback. It has been shown by Lindner ([6], see also [2]) that the additive functors on this category are precisely the Mackey functors for G.

2.10. It remains to keep track of identifications by Φ, i.e. to start with a morphism $f \in \mathrm{Hom}_{\mathbf{S}(G)}(X,Y)$, to lift it to

$$f^+ \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \ ,$$

and see when f^+ lies in $\mathrm{Ker}\,\phi$. Now f is represented by a difference of two G-sets over ${}_G(Y \times X)$ of the form

$$\begin{tikzcd} & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd} \quad - \quad \begin{tikzcd} & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X \end{tikzcd} .$$

By induction from $\Delta(G)$ to $G \times G$, the G-set on the left hand side lifts to the following $(G \times G)$-set over ${}_{(G\times G)}(Y \times G \times X)$

$$\begin{tikzcd} & G \times Z \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \arrow[dr, "\alpha"] & \\ Y & G & X \end{tikzcd}$$

where the $(G \times G)$-actions on $G \times Z$ and $Y \times G \times X$ are given respectively by $(s,t) \cdot (g,z) = (sgt^{-1}, tz)$ and $(s,t) \cdot (y,g,x) = (sy, sgt^{-1}, tx)$, and where

$$(\gamma, \beta, \alpha)(g,z) = \big(gb(z), g, a(z)\big) \ \ .$$

Similarly the G-set $\big(Z', (b', a')\big)$ lifts to $\big(G \times Z', (\gamma', \beta', \alpha')\big)$.

Now f^+ is in $\operatorname{Ker} \phi$ if and only if there is an isomorphism

$$\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & & & & G \times Z' \arrow[dl, "\gamma'"'] \arrow[dr, "\alpha'"] & \\ Y & & X & \xrightarrow{\theta} & Y & & X \end{tikzcd} \ .$$

of $(G \times G)$-sets over $Y \times X$. Since $(g,z) = g \cdot (1,z)$ for any $(g,z) \in G \times Z$, it follows that θ is a map from $G \times Z$ to $G \times Z'$ of the form

$$(g,z) \mapsto \big(gu(z), v(z)\big) \ \ ,$$

where u is a map from Z to G and v is a map from Z to Z'. Now for any $(s,t) \in G \times G$, the equality

$$\theta\big((s,t) \cdot (g,z)\big) = (s,t) \cdot \theta\big((g,z)\big)$$

gives

$$\big(sgt^{-1}u(tz), v(tz)\big) = \big(sgu(z)t^{-1}, tv(z)\big) \ \ .$$

This is equivalent to

$$u(tz) = {}^t u(z) \ \text{ and } \ v(tz) = tv(z) \ \ .$$

This means that u is a morphism of G-sets from Z to G^c, which is the set G with G-action by conjugation, and v is a morphism of G-sets.

Moreover θ is a bijection if and only if v is.

Finally θ is an morphism of (G,G)-bisets *over* $Y \times X$ if and only if $\alpha' \circ \theta = a$ and $\gamma' \circ \theta = \gamma$, i.e. equivalently if

$$a' \circ v = a \ \text{ and } \ gu(z) \cdot b' \circ v(z) = g \cdot b(z)$$

for any $(g,z) \in G \times Z$. In other words

$$a = a' \circ v \ \text{ and } \ b = u * (b' \circ v) \ \ ,$$

where, for any map $w : Z \to Y$, the map $u * w : Z \to Y$ is defined by $(u * w)(z) = u(z) \cdot w(z)$. The map $u * w$ is a map of G-sets if $u : Z \to G^c$

and $w : Z \to Y$ are. Note that $w' = u * w$ if and only if $w = \bar{u} * w'$, where $\bar{u} : Z \to G^c$ is defined by $\bar{u}(z) = u(z)^{-1}$.

It follows that f maps to the zero morphism in $\mathbf{B}(G)$ if and only if there exists $u : Z \to G^c$ and an isomorphism $v : Z \to Z'$ such that

$$a' \circ v = a \ \text{ and } \ b' \circ v = u * b \ \ ,$$

But then v is an isomorphism

$$\begin{tikzcd}
 & Z \arrow[dl, "b'\circ v"'] \arrow[dr, "a'\circ v"] & & & & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\
Y & & X & \xrightarrow{v} & Y & & X
\end{tikzcd} .$$

of G-sets over $Y \times X$, and f is also represented by the difference

$$\begin{tikzcd}
 & Z \arrow[dl, "b"'] \arrow[dr, "a"] & & & & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\
Y & & X & - & Y & & X
\end{tikzcd} ,$$

since $a' \circ v = a$ and $b' \circ v = u * b$. These are the morphisms in the category $\mathbf{S}(G)$ that vanish in $\mathbf{B}_\bullet(G)$. In other words:

2.11. Theorem : *Let G be a finite group. Let $\underline{\mathbf{S}}(G)$ denote the quotient category of $\mathbf{S}(G)$ defined by setting, for any two finite G-sets Y and Y*

$$\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X) \ \ ,$$

where $K(Y, X)$ is the subgroup generated by the differences

(2.12)
$$\begin{tikzcd}
 & Z \arrow[dl, "b"'] \arrow[dr, "a"] & & & & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\
Y & & X & - & Y & & X
\end{tikzcd} ,$$

where $a : Z \to X$, $b : Z \to Y$, and $u : Z \to G^c$ are morphisms of G-sets.

Then the functor $\Phi_\bullet$ induces an equivalence of categories $\underline{\mathbf{S}}(G) \cong \mathbf{B}_\bullet(G)$.

Since the difference 2.12 factors as

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "\mathrm{Id}"] & \\ Y & & Z \end{tikzcd} \circ \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ Z & & Z \end{tikzcd} - \begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ Z & & Z \end{tikzcd} \right) \circ \begin{tikzcd}[column sep=small] & Z \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "a"] & \\ Z & & X \end{tikzcd}$$

the morphism vanishing in $\underline{\mathbf{S}}(G)$ are generated in the category $\mathbf{S}(G)$ by the morphisms of the form

$$\begin{tikzcd} & Z \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ Z & & Z \end{tikzcd} - \begin{tikzcd} & Z \arrow[dl, "u*\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ Z & & Z \end{tikzcd} .$$

2.13. It follows that the additive functors from $\underline{\mathbf{S}}(G)$ to the category of abelian groups are exactly those Mackey functors (in the sense of Dress) such that for any G-set Z and any $u : Z \to G^c$, the morphism $M_*(u * \mathrm{Id})$ is equal to the identity map of $M(Z)$.

This condition is additive with respect to Z, since the map $u * \mathrm{Id}_Z$ maps each G-orbit of Z to itself. Hence these functors are exactly the functors for which the map $M_*(u*\mathrm{Id})$ is the identity map of $M(G/H)$, for any subgroup H of G and any $u : G/H \to G^c$. Such a map is of the form $gH \mapsto gcH$, where $c \in C_G(H)$. The map $u * \mathrm{Id} : G/H \to G/H$ is the map $gH \mapsto gcH$.

Translated in terms of the usual definition of Mackey functors, this map expresses the action of c on $M(H) = M(G/H)$. This shows that additive functors from $\underline{\mathbf{S}}(G)$ to abelian groups are exactly the Mackey functors for the group G such that, for any $H \leq G$, the centralizer $C_G(H)$ acts trivially on $M(H)$. These are the "conjugation invariant Mackey functors" introduced in [5].

3. Fused G-sets

Let Z be any (finite) G-set. The multiplication $(u, v) \mapsto u * v$ endows the set $\mathrm{Hom}_{G\text{-set}}(Z, G^c)$ with a group structure. Moreover, for any finite G-set X, this group acts on the left on the set $\mathrm{Hom}_{G\text{-set}}(Z, X)$, via $(u, f) \mapsto u * f$. This action is compatible with the composition of morphisms: if Y is a finite G-set, if $u : Z \to G^c$ and $v : Y \to G^c$ are morphisms of G-sets, then for any morphisms of G-sets $f : Z \to Y$ and $g : Y \to X$, one checks easily that

$$(v * g) \circ (u * f) = \big(u * (v \circ f)\big) * (g \circ f) \quad . \tag{3.1}$$

3.2. Notation : *Let G-$\underline{\textbf{set}}$ denote the category of* fused G-sets*: its objects are finite G-sets, and for any finite G-sets Z and Y*

$$\mathrm{Hom}_{G\text{-}\underline{\mathsf{set}}}(Z,Y)=\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z,G^c)\backslash \mathrm{Hom}_{G\text{-}\mathsf{set}}(Z,Y)\ \ .$$

The composition of morphisms in G-$\underline{\textbf{set}}$ is induced by the composition of morphisms in G-set.

3.3. Remark : For any G-set Y, set $Y^I=Y\times G^c$. This notation is chosen to evoke a path object in homotopy theory (cf. [4] Section 4.12). There is a natural morphism $p:Y^I\to Y\times Y$, defined by $p(y,g)=(y,gy)$, for $y\in Y$ and $g\in G$, and a morphism $i:Y\to Y^I$ defined by $i(y)=(y,1)$, for $y\in Y$. The composition $p\circ i$ is equal to the diagonal map $Y\to Y\times Y$.

Two morphisms $a,b:Z\to Y$ in G-set are equal in the category G-$\underline{\textbf{set}}$ if and only if the morphism $(a,b):Z\to Y\times Y$ factors as

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ Z \arrow[ur, "\varphi"] \arrow[r, "(a,b)"'] & Y\times Y \end{tikzcd}$$

for some morphism of G-sets $\varphi:Z\to Y^I$.

3.4. Remark : It follows from 3.1 that the map $u\mapsto u*\mathrm{Id}_Z$ is a group antihomomorphism from $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z,G^c)$ to the group of G-automorphisms of Z. Hence a morphism $\underline{f}:Z\to Y$ in the category G-$\underline{\textbf{set}}$ is an isomorphism if and only if any of its representatives $f:Z\to Y$ in G-set is an isomorphism.

3.5. Weak pullbacks of fused G-sets. Disjoint union of G-sets is a coproduct in G-$\underline{\textbf{set}}$. There is also a weak version of pullback in G-$\underline{\textbf{set}}$: let

$$\begin{tikzcd} & T \arrow[dl, "\underline{c}"'] \arrow[dr, "\underline{d}"] & \\ X \arrow[dr, "\underline{a}"'] & & Y \arrow[dl, "\underline{b}"] \\ & Z & \end{tikzcd}$$

be a commutative diagram in G-$\underline{\textbf{set}}$, where underlines denote the images in G-$\underline{\textbf{set}}$ of morphisms in G-set. This means that $\underline{a}\circ\underline{c}=\underline{b}\circ\underline{d}$, i.e. that there

exists $u \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, G^c)$ such that

$$b \circ d = u * (a \circ c) \ \ .$$

But $u * (a \circ c) = a \circ (u * c)$. It follows that there is a unique morphism $e \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, X \times_{a,b} Y)$ such that the diagram

$$\begin{tikzcd}
 & T \arrow[d, dashed, "e"'] \arrow[ddl, "u*c"'] \arrow[ddr, "d"] & \\
 & X \times_{a,b} Y \arrow[dl, "p"] \arrow[dr, "q"'] & \\
X \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\
 & Z &
\end{tikzcd}$$

is commutative in G-set, where $p : X \times_{a,b} Y \to X$ and $q : X \times_{a,b} Y \to Y$ are the canonical morphisms from the pullback $X \times_{a,b} Y$. In other words, the diagram

(3.6)

$$\begin{tikzcd}
 & T \arrow[d, "\underline{e}"'] \arrow[ddl, "\underline{c}"'] \arrow[ddr, "\underline{d}"] & \\
 & X \times_{a,b} Y \arrow[dl, "\underline{p}"] \arrow[dr, "\underline{q}"'] & \\
X \arrow[dr, "\underline{a}"'] & & Y \arrow[dl, "\underline{b}"] \\
 & Z &
\end{tikzcd}$$

is commutative in G-$\underline{\mathbf{set}}$.

But still $(X \times_{a,b} Y, \underline{p}, \underline{q})$ need not be a pullback in G-$\underline{\mathbf{set}}$, since the morphism $\underline{e}$ making Diagram 3.6 commutative is generally not unique, as e itself depends on the choice of u. Moreover, the lifts a and b of $\underline{a}$ and $\underline{b}$ to G-set are not unique : it should be noted however that if $a' = v * a$ and $b' = w * b$ are other lifts of a and b, respectively, where $v \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(X, G^c)$ and $w \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(Y, G^c)$, then the map $f : (x, y) \mapsto \big(v(x)x, w(y)y\big)$ is an

isomorphism of G-sets from $X \times_{a',b'} Y$ to $X \times_{a,b} Y$, such that the diagram

$$\begin{tikzcd}
& X \times_{a',b'} Y \arrow[dl, "p'"'] \arrow[dr, "q'"] \arrow[drrr, "f"] & & & & \\
X \arrow[dr, "a'"'] \arrow[drrr, "v*\mathrm{Id}"' near end] & & Y \arrow[dl, "b'"] \arrow[drrr, "w*\mathrm{Id}" near end] & & X \times_{a,b} Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\
& Z \arrow[drrr, "\mathrm{Id}"'] & & X \arrow[dr, "a"] & & Y \arrow[dl, "b"] \\
& & & & Z &
\end{tikzcd}$$

is commutative in G-set. Since $\underline{a'} = \underline{a}$, $\underline{b'} = \underline{b}$, $\underline{v * \mathrm{Id}} = \underline{\mathrm{Id}}$, and $\underline{w * \mathrm{Id}} = \underline{\mathrm{Id}}$, this yields a commutative diagram

$$\begin{tikzcd}
& X \times_{a',b'} Y \arrow[r, "\underline{f}"] \arrow[dl, "\underline{p'}"'] \arrow[drr, "\underline{q'}" near end] & X \times_{a,b} Y \arrow[dll, "\underline{p}"' near end] \arrow[dr, "\underline{q}"] & \\
X \arrow[dr, "\underline{a}"'] & & & Y \arrow[dll, "\underline{b}"] \\
& Z & &
\end{tikzcd}$$

in G-**set**, and $\underline{f}$ is an isomorphism. This shows that the weak pullback $X \times_{a,b} Y$ only depends on $\underline{a}$ and $\underline{b}$ in the category G-**set**. For this reason, it may be denoted by $X \times_{\underline{a},\underline{b}} Y$.

3.7. Spans of fused G-sets. Recall (cf. [9], [1] for the general definition) that if X and Y are finite G-sets, then a *span* $\Lambda_{Z,\underline{a},\underline{b}}$ over X and Y in the category G-**set** is a diagram of the form

$$\begin{tikzcd}
& Z \arrow[dl, "\underline{a}"'] \arrow[dr, "\underline{b}"] & \\
X & & Y
\end{tikzcd}$$

where Z is a finite G-set and $\underline{a}, \underline{b}$ are morphisms in the category G-**set**. Two spans $\Lambda_{Z,\underline{a},\underline{b}}$ and $\Lambda_{Z',\underline{a'},\underline{b'}}$ over X and Y are equivalent if there exists an isomorphism $\underline{f} : Z \to Z'$ in G-**set** such that the diagram

$$\begin{tikzcd}
& Z \arrow[dl, "\underline{a}"'] \arrow[dr, "\underline{b}"] \arrow[dd, "\underline{f}"] & \\
X & & Y \\
& Z \arrow[ul, "\underline{a'}"] \arrow[ur, "\underline{b'}"'] &
\end{tikzcd}$$

is commutative. The set of equivalence classes of spans of fused G-sets over X and Y is an additive monoid, where the addition is defined by disjoint union (i.e. $\Lambda_{Z_1,\underline{a}_1,\underline{b}_1} + \Lambda_{Z_2,\underline{a}_2,\underline{b}_2} = \Lambda_{Z_1 \sqcup Z_2,\underline{a}_1 \sqcup \underline{a}_2,\underline{b}_1 \sqcup \underline{b}_2}$). The corresponding Grothendieck group is isomorphic to $\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(Y,X)$.

It should be noted that even if there is no pullback construction in the category G-**set**, the isomorphism classes of spans in G-**set** can still be composed by *weak pullback*, and this induces the composition of morphisms in $\underline{\mathbf{S}}(G)$.

4. Fused Mackey functors

4.1. Definition : *Let R be a commutative ring. Let $R\,\mathbf{S}(G)$ (resp. $R\,\underline{\mathbf{S}}(G)$) denote the R-linear extension of the category $\mathbf{S}(G)$ (resp. $\underline{\mathbf{S}}(G)$), defined as follows:*

- *The objects of $R\,\mathbf{S}(G)$ and $R\,\underline{\mathbf{S}}(G)$ are finite G-sets.*
- *For finite G sets X and Y,*

$$\mathrm{Hom}_{R\,\mathbf{S}(G)}(X,Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{S}(G)}(X,Y) \ ,$$

$$\mathrm{Hom}_{R\,\underline{\mathbf{S}}(G)}(X,Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X,Y) \ .$$

- *Composition of morphisms is induced by the pullback in G-set (resp. the weak pullback in G-**set**).*

A Mackey functor for G over R in the sense of Lindner ([6]) is an R-linear functor from $R\,\mathbf{S}(G)$ to the category R-Mod of R-modules.

Similarly, a fused Mackey functor *for G over R is an R-linear functor from $R\,\underline{\mathbf{S}}(G)$ to R-Mod. A morphism of fused Mackey functors is a natural transformation of functors. Fused Mackey functors for G over R form a category denoted by $\mathsf{Mack}_R^f(G)$.*

The following is an equivalent definition of fused Mackey functors, *à la* Dress:

4.2. Definition : *Let R be a commutative ring. A* fused Mackey functor *for the group G over R is a bivariant R-linear functor $M = (M^*, M_*)$ from G-**set** to R-Mod such that:*

1. *For any finite G-sets X and Y, the maps*

$$M(X) \oplus M(Y) \underset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\overset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\rightleftarrows}} M(X \sqcup Y)$$

   induced by the canonical inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ are mutual inverse isomorphisms.

2. *If*

$$\begin{tikzcd} & X \times_{\underline{a},\underline{b}} Y \arrow[dl, "\underline{p}"] \arrow[dr, "\underline{q}"] & \\ X \arrow[dr, "\underline{a}"] & & Y \arrow[dl, "\underline{b}"] \\ & Z & \end{tikzcd}$$

   *is a weak pullback diagram in G-**set**, then $M^*(\underline{a})M_*(\underline{b}) = M_*(\underline{p})M^*(\underline{q})$.*

A morphism of fused Mackey functors is a natural transformation of bivariant functors.

The category $\mathsf{Mack}_R^f(G)$ can be viewed as a full subcategory of the category $\mathsf{Mack}_R(G)$ of Mackey functors for G over R. In the case $R = \mathbb{Z}$, this category is equivalent to the category of conjugation invariant Mackey functors introduced in [5].

The inclusion functor $\mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G)$ has a left adjoint:

4.3. Definition : *Let M be a Mackey functor for G over R, in the sense of Lindner, i.e. an R-linear functor $R\mathbf{S}(G) \to R$-Mod. When X is a finite G-set, set*

$$M^f(X) = M(X) / \sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big) \ ,$$

where the summation runs through triples (Z, a, u) consisting of a finite G-set Z, and morphisms of G-sets $a : Z \to X$ and $u : Z \to G^c$, and $\Lambda_{a,\mathrm{Id}_Z}$ denotes the span

$$\begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[dr, equal, "\mathrm{Id}_Z"] & \\ X & & Z \end{tikzcd}$$

of G-sets.

4.4. Proposition : *Let R be a commutative ring, and G be a finite group.*

1. *Let M be a Mackey functor for G over R. The correspondence*

$$X \mapsto M^f(X)$$

is a fused functor M^f for G over R.

2. *The correspondence $\mathcal{F} : M \mapsto M^f$ is a functor from $\mathsf{Mack}_R(G)$ to $\mathsf{Mack}_R^f(G)$, which is left adjoint to the inclusion functor*

$$\mathcal{I} : \mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G) \ \ .$$

Moreover $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor of $\mathsf{Mack}_R^f(G)$.

Proof : For Assertion 1, to prove that M^f is a Mackey functor, observe that if $\Lambda_{Z,a,b}$ is a span of finite G-sets of the form

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X & & Y \end{tikzcd}$$

and $u : Z \to G^c$ is a morphism of G-sets, then

$$\Lambda_{Z,a,b} - \Lambda_{Z,u*a,b} = (\Lambda_{Z,a,\mathrm{Id}_Z} - \Lambda_{Z,u*a,\mathrm{Id}_Z}) \circ \Lambda_{Z,\mathrm{Id}_Z,b} \ \ .$$

It follows that the R-module

$$\sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big)$$

is equal to the sum

$$\sum_{Z,a,b,u} \mathrm{Im}\big(M(\Lambda_{a,b}) - M(\Lambda_{u*a,b})\big) \ \ .$$

In other words, it is equal to the image by M of the R-submodule $K_R(X,Y)$ of $\mathrm{Hom}_{R\mathbf{S}(G)}(Y,X)$ generated by the morphisms $\Lambda_{a,b} - \Lambda_{u*a,b}$, i.e. to the kernel of the quotient morphism

$$\mathrm{Hom}_{R\mathbf{S}(G)}(Y,X) \to \mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(Y,X) \ \ .$$

This shows that K_R is an ideal in the category $R\,\mathbf{S}(G)$. So if M is an R-linear functor $R\,\mathbf{S}(G) \to R\text{-}\mathsf{Mod}$, the correspondence

$$X \mapsto M^f(X) = M(X)/\sum_{f\in K_R(X,Y)} \mathrm{Im}M(f)$$

is an R-linear functor from the quotient category $R\,\underline{\mathbf{S}}(G)$ to R-Mod.

Assertion 2 is straightforward: first it is clear that $\mathcal{F}\circ\mathcal{I}$ is isomorphic to the identity functor, since $N^f=N$ when N is a fused Mackey functor. This isomorphism $\mathcal{F}\circ\mathcal{I}\cong \mathrm{Id}_{\mathsf{Mack}_R^f(G)}$ provides the counit of the adjunction. Next for any Mackey functor M, there is a projection morphism $M\to\mathcal{I}\mathcal{F}(M)$, and this yields the unit of the adjunction. □

4.5. Remark : Assertion 2 shows that $\mathsf{Mack}_R^f(G)$ is a *reflective* subcategory of $\mathsf{Mack}_R^f(G)$ (cf. [7], Chapter IV, Section3).

4.6. Remark : If the Mackey functor M is given in the sense of Dress, then for any finite G-set X

$$M^f(X)=M(X)/\sum_{\substack{a:Z\to X\\ u:Z\to G^c}}\mathrm{Im}\big(M_*(a)-M_*(u*a)\big)\ \ ,$$

where Z is a finite G-set, and a,u are morphisms of G-sets.

4.7. Corollary :

1. *If P is a projective Mackey functor, then P^f is projective in the category* $\mathsf{Mack}_R^f(G)$.
2. *The category* $\mathsf{Mack}_R^f(G)$ *has enough projective objects. More precisely, if N is a fused Mackey functor, and $\theta: P\to\mathcal{I}(N)$ is an epimorphism in* $\mathsf{Mack}_R(G)$ *from a projective Mackey functor P, then $\mathcal{F}(\theta):P^f\to N$ is an epimorphism in* $\mathsf{Mack}_R^f(G)$.

Proof : Assertion 1 follows from the fact that $\mathcal{F}$ is left adjoint to the exact functor $\mathcal{I}$. Assertion 2 is then straightforward. □

5. The fused Mackey algebra

When G is a finite group, set $\Omega_G=\bigsqcup_{H\leq G}G/H$, and let RB_{Ω_G} denote the Dress construction for the Burnside functor RB over the ring R. Recall that RB_{Ω_G}, as a Mackey functor in the sense of Dress, is obtained by precomposition of RB with the endofunctor $X\mapsto X\times\Omega_G$ of G-set.

Also recall (cf. [2] Lemma 7.3.2 and Proposition 4.5.1) that the functor RB_{Ω_G} is a progenerator of the category $\mathsf{Mack}_R(G)$, and that the algebra $\mathrm{End}_{\mathsf{Mack}_R(G)}(B_{\Omega_G})\cong B(\Omega_G^2)$ is isomorphic to the Mackey algebra $\mu_R(G)$ of G over R, introduced by Thévenaz and Webb ([8]).

It follows from Corollary 4.7 that the functor $(RB_{\Omega_G})^f$ is a progenerator in the category $\mathsf{Mack}_R^f(G)$. Hence this category is equivalent to the category of modules over the algebra $\mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big)$.

5.1. Definition : *The* fused Mackey algebra *of G over R is the algebra*

$$\mu_R^f(G) = \mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big) \quad .$$

5.2. Lemma : *Let X be a finite G-set. Then $(RB_X)^f$ is isomorphic to the Yoneda functor* $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X,-)$.

Proof : Denote by $\mathcal{Y}_X$ the Yoneda functor $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X,-)$. For any fused Mackey functor N for G over R

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Mack}_R^f(G)}\big((RB_X)^f,N\big) &\cong \mathrm{Hom}_{\mathsf{Mack}_R(G)}\big(RB_X,\mathcal{I}(N)\big)\\ &\cong \mathcal{I}(N)(X) \cong N(X)\\ &\cong \mathrm{Hom}_{\mathsf{Mack}_R^f(G)}(\mathcal{Y}_X,N) \quad .\end{aligned}$$

The lemma follows, since all these isomorphisms are natural. $\square$

5.3. Theorem : *The fused Mackey algebra $\mu_R^f(G)$ is isomorphic to the quotient of the algebra $RB(\Omega_G^2) \cong \mu_R(G)$ by the R-module generated by differences of the form*

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ \Omega_G & & \Omega_G\end{tikzcd} \quad - \quad \begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ \Omega_G & & \Omega_G\end{tikzcd} ,$$

where $a,b : Z \to \Omega_G$ and $u : Z \to G^c$ are morphisms of G-sets.

Proof : This follows from Lemma 5.2, since the quotient in the theorem is precisely $\mathrm{End}_{R\underline{\mathbf{S}}(G)}(\Omega_G)$. $\square$

5.4. Remark : One can deduce from this theorem that the fused Mackey algebra $\mu_R^f(G)$ is always free of finite rank as an R-module, and this rank does not depend on the commutative ring R. More precisely, Thévenaz and Webb have shown ([8] Proposition 3.2) that the Mackey algebra $\mu_R(G)$ has

an R-basis consisting of elements of the form

$$t_K^H \, c_{g,K} \, r_{K^g}^L \quad ,$$

where (H, L, g, K) runs through a set of representatives of 4-tuples consisting of two subgroups H and L of G, and element g of G, and a subgroup K of $H \cap {}^gL$, for the equivalence relation $\equiv$ given by

$$(H, L, g, K) \equiv (H', L', g', K') \Leftrightarrow \begin{cases} H = H', \; L = L', \\ \text{and} \\ \exists h \in H, \; \exists l \in L, \; g' = hgl, \; K' = {}^hK \quad . \end{cases}$$

Similarly, the quotient algebra $\mu_R^f(G)$ of $\mu_R(G)$ has a basis consisting of the images of the elements $t_K^H \, c_{g,K} \, r_{K^g}^L$, where (H, L, g, K) runs through a set of representatives of 4-tuples as above, modulo the relation $\equiv^f$ defined by

$$(H, L, g, K) \equiv^f (H', L', g', K') \Leftrightarrow \begin{cases} H = H', \; L = L', \\ \text{and} \\ \exists h \in H, \; \exists l \in L, \; \exists x \in C_G(K), \\ \quad g' = hxgl, \; K' = {}^hK \quad . \end{cases}$$

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